The Regularization of a Logic

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Contents

Logics as substitution invariant consequence relations

Definition

Let $\boldsymbol{Fm}$ the term algebra built up with countably many variables. A logic is a consequence relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$, which is substitution-invariant in the sense that for every substitution $\sigma \colon \boldsymbol{Fm} \to \boldsymbol{Fm}$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma\Gamma \vdash \sigma\varphi.$$

Matrices

Definition

1. A (logical) matrix is a pair $\langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A$ (F is called the filter of the matrix).
2. Every class of matrices M induces a logic as follows:

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$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \boldsymbol{A}, F\rangle \in \mathsf{M} \text{ and hom } v\colon \boldsymbol{Fm} \to \boldsymbol{A}$$
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Example

- Let K be the variety of Boolean algebras. Define $\vdash_{\mathsf{K}}$ as the logic induced by the following class of matrices:

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{K}, F \text{ is a lattice filter on } \boldsymbol{A}\}.$$

$\vdash_{\mathsf{K}}$ coincides with $\vdash_{\mathsf{CL}}$.

Matrices as models of logics

Definition

Let $\vdash$ be a logic.

- A matrix $\langle \boldsymbol{A}, F \rangle$ is a model of a logic $\vdash$ when

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- $\vdash$ is complete with respect to a class of matrices M when $\vdash = \vdash_{\mathsf{M}}$.
- We set $\mathrm{Mod}(\vdash) \coloneqq \{\langle \boldsymbol{A}, F\rangle : \langle \boldsymbol{A}, F\rangle \text{ is a model of } \vdash\}$.

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- $\vdash$ is complete with respect to a class of matrices M when $\vdash = \vdash_{\mathsf{M}}$.
- We set $\mathrm{Mod}(\vdash) := \{\langle \boldsymbol{A}, F \rangle : \langle \boldsymbol{A}, F \rangle \text{ is a model of } \vdash\}$.

- Remark: $\mathrm{Mod}(\vdash)$ is a very artificial class of matrices, since

$$\langle \boldsymbol{A}, A \rangle \in \mathrm{Mod}(\vdash) \text{ for every algebra } \boldsymbol{A}.$$

Leibniz and Suszko congruence

- We need a process to tame the matrices in $\mathrm{Mod}(\vdash)$:

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Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$.

1. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is compatible with F when

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta, \text{ then } b \in F.$$

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3. The Suszko congruence of F (over $\boldsymbol{A}$) is $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}} F := \{\bigcap \boldsymbol{\Omega} G : G \supseteq F \text{ and } G \text{ is a filter}\}$

Reduced models

Definition

Let $\vdash$ be a logic.

1. The class of Leibniz reduced models of $\vdash$ is

$$\mathrm{Mod}^{*}(\vdash) := \{\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash) : \boldsymbol{\Omega}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}\}$$

2. The class of Suszko reduced models of $\vdash$ is

$$\mathrm{Mod}^{\mathrm{Su}}(\vdash) := \{\langle \boldsymbol{A}, F\rangle \in \mathrm{Mod}(\vdash) : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}} F = \mathrm{Id}_{\boldsymbol{A}}\}$$

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- In most cases, reduced models (as opposed to arbitrary models) of a logic are its intended matrix semantics.

Direct systems of algebras

Definition

A direct system of algebras consists in

1. A join-semilattice $I = \langle I, \vee \rangle$;
2. A family of algebras $\{\boldsymbol{A}_i : i \in I\}$ with disjoint universes;
3. A homomorphism $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$, for every $i, j \in I$ such that $i \leq j$

such that f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Płonka sums over a direct system of algebras

Definition

Let X be a direct system of algebras. The *Płonka sum* over X is a new algebra $\mathcal{P}_I(X)$ s.t.

1. the universe of $\mathcal{P}_I(X) = \bigcup_{i \in I} A_i$
2. for every n-ary basic operation f on $\boldsymbol{A}_i$ and $a_1, \ldots, a_n \in \bigcup_{i \in I} A_i$, we set
$$f^{\mathcal{P}_I(\boldsymbol{A}_i)_{i \in I}}(a_1, \ldots, a_n) := f^{\boldsymbol{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n))$$
where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

Płonka sums over a direct system of algebras

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Let X be a direct system of algebras. The *Płonka sum* over X is a new algebra $\mathcal{P}_l(X)$ s.t.

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where $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$ and $j = i_1 \vee \dots \vee i_n$.

Example!

Płonka sums over a direct system of algebras

We extend the previous definitions to logical matrices as follows.

Definition

A *direct system* of matrices consists in

1. A join-semilattice $I = \langle I, \vee \rangle$.
2. A family of matrices $\{\langle \boldsymbol{A}_i, F_i \rangle : i \in I\}$.
3. A homomorphism $f_{ij} \colon \boldsymbol{A}_i \to \boldsymbol{A}_j$ such that $f_{ij}[F_i] \subseteq F_j$, for every $i, j \in I$ such that $i \leq j$.

Given directed system of matrices X as above, we set

$$\mathcal{P}_I(X) := \langle \mathcal{P}_I(\boldsymbol{A}_i)_{i \in I}, \bigcup_{i \in I} F_i \rangle.$$

The matrix $\mathcal{P}_I(X)$ is the *Płonka sum* of the direct system of matrices X. Given a class M of matrices, we denote by $\mathcal{P}_I(\mathsf{M})$ the class of all Płonka sums of directed systems of matrices in M.

Contents

Definition and examples

Definition

Let $\vdash$ be a logic. The logic of variable inclusion of $\vdash$ (or its *regularization*) is the relation $\vdash^r \subseteq \mathcal{P}(Fm) \times Fm$ defined as follows:

$$\Gamma \vdash^r \varphi \iff \text{there is } \Gamma' \subseteq \Gamma \text{ s.t. } Var(\Gamma') \subseteq Var(\varphi) \text{ and } \Gamma' \vdash \varphi.$$

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Example

Let $\vdash_{\mathcal{CL}}$ be classical logic. Its logic of variable inclusion $\vdash^r_{\mathcal{CL}}$ is the logic $\vdash_{\mathcal{PWK}}$ known as Paraconsistent Weak Kleene logic.

Models

Lemma (Soundness)

Let $\vdash$ be a logic and X be a direct systems of models of $\vdash$. Then $\mathcal{P}_I(X)$ is a model of $\vdash^r$.

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Let $\vdash$ *be a logic and* X *be a direct systems of models of* $\vdash$. *Then* $\mathcal{P}_I(X)$ *is a model of* $\vdash^r$.

Theorem (Completeness)

Let $\vdash$ *be a logic and* M *be a class of matrices containing the matrix* $\langle \mathbf{1}, \{1\} \rangle$. *If* $\vdash$ *is complete w.r.t.* M, *then* $\vdash^r$ *is complete w.r.t.* $\mathcal{P}_I(\mathsf{M})$.

Models

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Let $\vdash$ *be a logic and* X *be a direct systems of models of* $\vdash$*. Then* $\mathcal{P}_l(X)$ *is a model of* $\vdash^r$*.*

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Corollary

Let $\vdash$ *be a logic. Its logic of variable inclusion* $\vdash^r$ *is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_l(\mathsf{Mod}(\vdash)) \quad \mathcal{P}_l(\mathsf{Mod}^*(\vdash)) \quad \mathcal{P}_l(\mathsf{Mod}^{Su}(\vdash)).$$

Logics with a partition function

Definition (Essentially Płonka)

A logic $\vdash$ has a partition function is there is a formula $\pi(x, y)$ in which the variables x and y really occur such that $x \vdash \pi(x, y)$ and the following equations hold in $\{\boldsymbol{A} : \exists F \subseteq A \text{ s.t. } \langle \boldsymbol{A}, F \rangle \in \mathrm{Mod}^{\mathrm{Su}}(\vdash)\}$ for every n-ary connective f:

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1. $\boldsymbol{A} \vDash \pi(x,x) \approx x$
2. $\boldsymbol{A} \vDash \pi(\pi(x,y),z) \approx \pi(x,\pi(y,z))$
3. $\boldsymbol{A} \vDash \pi(x,\pi(y,z)) \approx \pi(x,\pi(z,y))$
4. $\boldsymbol{A} \vDash \pi(f(x_1,\dots,x_n),y) \approx f(\pi(x_1,y),\dots,\pi(x_n,y))$
5. $\boldsymbol{A} \vDash \pi(f(x_1,\dots,x_n),x_i) \approx f(x_1,\dots,x_n), i \in \{1,\dots,n\}$
6. $\boldsymbol{A} \vDash \pi(y,f(x_1,\dots,x_n)) \approx \pi(y,f(\pi(y,x_1),\dots,\pi(y,x_n)))$.
7. $\boldsymbol{A} \vDash \pi(x,f(x,\dots,x)) \approx x$.

Note that in every logic with a lattice reduct the term $\pi(x,y) = x \wedge (x \vee y)$ is a partition function!

Hilbert-style axiomatisation

Definition

Let $\mathcal{H}$ be a Hilbert-style calculus with finite rules, which determines a logic $\vdash$ with a partition function π. Let $\mathcal{H}^r$ be the Hilbert-style calculus given by the following rules:

$$\emptyset \rhd \psi \tag{1}$$

$$\gamma_1, \ldots, \gamma_n \rhd \pi(\varphi, \pi(\gamma_1, \pi(\gamma_2, \ldots, \pi(\gamma_{n-1}, \gamma_n) \ldots))) \tag{2}$$

$$x \rhd \pi(x, y) \tag{3}$$

$$\chi(\delta, \vec{z}) \lhd \rhd \chi(\epsilon, \vec{z}) \tag{4}$$

for every

(i) $\emptyset \rhd \psi$ rule in $\mathcal{H}$;

(ii) $\gamma_1, \ldots, \gamma_n \rhd \varphi$ rule in $\mathcal{H}$;

(iii) $\epsilon \approx \delta$ equation in the definition of partition function, and formula $\chi(v, \vec{z})$.

Hilbert-style axiomatization

Theorem

Let $\vdash$ be a logic defined by a Hilbert-style calculus with finite rules $\mathcal{H}$. Then $\mathcal{H}^r$ is a complete Hilbert-style calculus for $\vdash^r$.

Example

Hilbert-calculus for $\vdash_{\mathsf{CL}}$:

1. $\rhd x \to (y \to x)$
2. $\rhd x \to (y \to z) \to ((x \to y) \to (x \to z))$
3. $\rhd (x \to y) \to (\neg y \to \neg x)$
4. $x, x \to y \rhd y$

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3. $\rhd (x \to y) \to (\neg y \to \neg x)$
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Hilbert-calculus for $\vdash^r_{\mathsf{CL}} = \vdash_{\mathsf{PWK}}$:

- (1-3) as axioms
- $x, x \to y \rhd y \wedge (y \vee (x \wedge (x \vee x \to y)))$
- $x \rhd x \wedge (x \vee y)$
- $\chi(\delta, \vec{z}) \lhd \rhd \chi(\epsilon, \vec{z})$.

Suszko reduced models

Theorem

Let $\vdash$ be a logic with a partition function π, and let X be a directed system of matrices in $\mathrm{Mod}^{Su}(\vdash)$. *TFAE:*

1. $\mathcal{P}_I(X) \in \mathrm{Mod}^{Su}(\vdash^r)$.
2. *For every $n, i \in I$ such that $\langle \boldsymbol{A}_n, F_n \rangle$ is trivial and $n < i$, there exists $j \in I$ s.t. $n \leq j, i \nleq j$ and $\boldsymbol{A}_j$ is non-trivial.*

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The theorem identifies the Suszko reduced models of $\vdash^r$, which can be expressed in terms of Płonka sums of Suszko reduced models of $\vdash$. Is it true that all Suszko models of $\vdash^r$ are of this kind? In general, the answer is no.

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Further results

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- Full characterisation of $\mathsf{Mod}^{\mathsf{Su}}(\vdash^r)$ for $\vdash$ finitary and equivalential
- Classification of $\vdash^r$ within the Leibniz Hierarchy
- Algebraizability of Gentzen systems associated with $\vdash^r$

Thank you!